Institute for Logic, Language and Computation

A Bimodal Provability Logic – Towards an Interpretability Logic with a Supremum Operator

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Interpretability

Example: ZFC $\rhd$ PA

$T \rhd S$: $S \vdash A \Rightarrow T \vdash A^{\tau}$, for some structure–preserving translation τ

Two perspectives on interpretability:

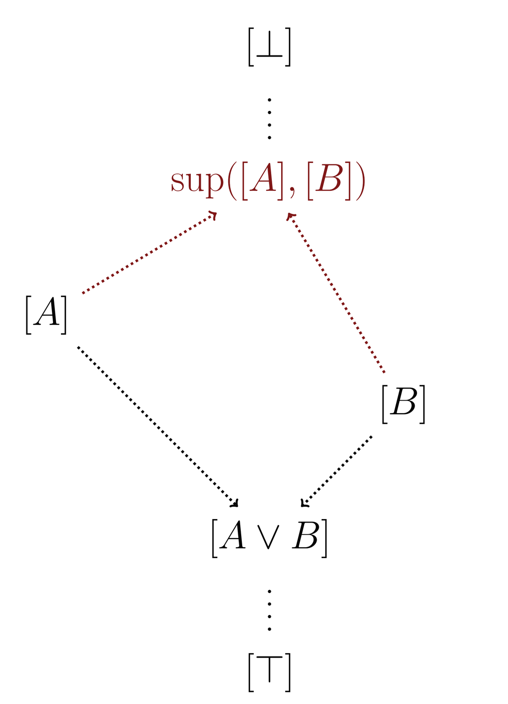
Notion for comparing theories ⇓ | Generalisation of provability ⇓

Lattice of Degrees

Degree of T: $\{S \mid S \rhd T \ \&\ T \rhd S\}$

Theorem (Švejdar): The degrees of finite extensions of Peano Arithmetic (PA) form a lattice.

$[A]$: the degree of $\mathsf{PA} + A$



Interpretability Logic

Modal Language	*Meaning in* $\mathcal{L}_{\mathsf{PA}}$
$\Box A$:	A is provable in PA
$A \rhd B$:	$\mathsf{PA} + A$ interprets $\mathsf{PA} + B$

Axioms of ILM:

- L1 $\Box(A \to B) \to (\Box A \to \Box B)$
- L2 $\Box(\Box A \to A) \to \Box A$
- J1 $\Box(A \to B) \to A \rhd B$
- J2 $(A \rhd B) \wedge (B \rhd C) \to (A \rhd C)$
- J3 $(A \rhd C) \wedge (B \rhd C) \to (A \vee B) \rhd C$
- J4 $A \rhd B \to (\Diamond A \to \Diamond B)$
- J5 $\Diamond A \rhd A$
- M $A \rhd B \to (A \wedge \Box C) \rhd (B \wedge \Box C)$

Rules: modus ponens, necessitation for $\Box$

Theorem (Shavrukov, Berarducci): $\vdash_{\mathsf{ILM}} A \Leftrightarrow \vdash_{\mathsf{PA}} A^*$ for all translations A^* of A into the language $\mathcal{L}_{\mathsf{PA}}$ of arithmetic

Aim of Current Work

$\vdash_{\mathsf{ILM}}$ "$A \vee B$ is an infimum of A and B in the lattice of degrees."

On the other hand:

Fact: There exist A, B with $[A \wedge B] \neq \mathrm{sup}([A],[B])$.

Fact: The existence of $\mathrm{sup}([A],[B])$ is *not* expressible in $\mathcal{L}_{\mathsf{ILM}}$.

⇛ add to ILM a new modality whose arithmetical interpretation is the supremum

First Approach

Add to ILM a binary modality $\oslash$, together with the axiom

$$C \rhd A \ \wedge\ C \rhd B \quad \leftrightarrow \quad C \rhd A \oslash B.$$

Intended arithmetical meaning of $A \oslash B$: a sentence in $\mathrm{sup}([A],[B])$.
An example (Švejdar): θ with

$$\vdash_{\mathsf{PA}} \theta \leftrightarrow \forall x(\Diamond_x \theta \to \Diamond_x A \wedge \Diamond_x B),$$

where $\Box_0$ stands for provability in $\mathsf{I}\Delta_0 + \mathsf{exp}$, and $\Box_{n+1}$ for provability in $\mathsf{I}\Sigma_{n+1}$.

A Discovery

Shavrukov: there is a formula $\triangledown$ of $\mathcal{L}_{\mathsf{PA}}$ with one free variable, such that for all sentences A, B, and C of $\mathcal{L}_{\mathsf{PA}}$,

$$\vdash_{\mathsf{PA}} C \rhd A \ \wedge\ C \rhd B \quad \leftrightarrow \quad C \rhd \triangledown A \wedge \triangledown B$$

⇛ it suffices to add a *unary* modality $\triangledown$ to ILM

A Nonstandard Provability Predicate

Let $\triangle A := \neg \triangledown \neg A$. Then

$$\triangle A = \exists x\,(\Box_x A \wedge \neg \Box^{\Pi_1}_{x-1} \bot)$$

where $\Box^{\Pi_1}_{n+1}$ ($\Box^{\Pi_1}_0$) stands for provability in $\mathsf{I}\Sigma_{n+1}$ ($\mathsf{I}\Delta_0 + \mathsf{exp}$) with a Π_1–oracle.

⇛ $\triangle$ expresses a strong notion of provability, call it s–*provability*

The axioms and rules of provability logic (GL) are valid for s–provability in PA:

1. $\vdash_{\mathsf{PA}} A \ \Rightarrow\ \vdash_{\mathsf{PA}} \triangle A$
2. $\vdash_{\mathsf{PA}} \triangle(A \to B) \to (\triangle A \to \triangle B)$
3. $\vdash_{\mathsf{PA}} \triangle(\triangle A \to A) \to \triangle A$

A Bimodal Provability Logic

Interpretation # 1
$\Box A$: A is provable in PA
$\triangle A$: A is s-provable

GLT, a first step towards adding $\triangle$ to ILM:

- T1 $\triangle(A \to B) \to (\triangle A \to \triangle B)$
- T2 $\triangle(\triangle A \to A) \to \triangle A$
- T3 $\Box(A \to B) \to (\Box A \to \Box B)$
- T4 $\triangle A \to \Box A$
- T5 $\Box A \to \triangle \Box A$
- T6 $\Box A \to \Box \triangle A$
- T7 $\Box \triangle A \to \Box A$

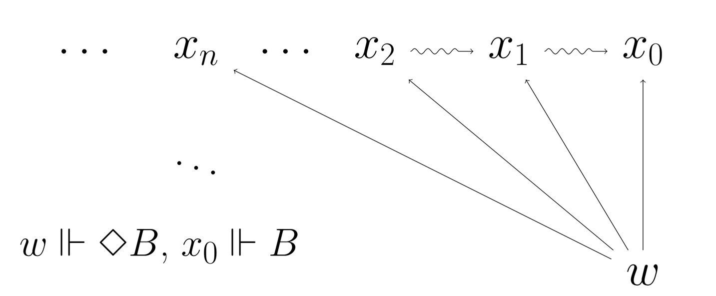
Rules: modus ponens, and necessitation for $\triangle$

Interpretation # 2
Parikh's rule: from $\Box A$, infer A.
$\Box A$: A is provable in PA + Parikh's rule
$\triangle A$: A is provable in PA

Standard Semantics

Model: $\langle W, Q, R, \Vdash \rangle$ with

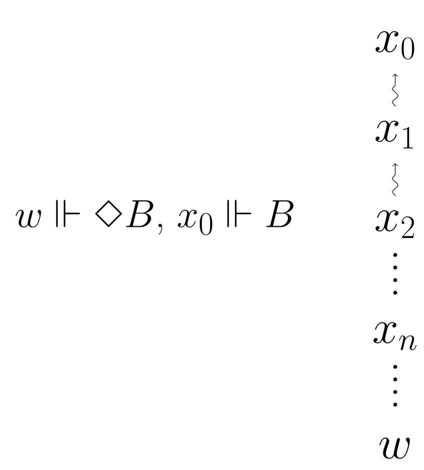
1. R, Q conversely well–founded & transitive
2. $R \subseteq Q$
3. $Q \circ R \subseteq R$, $R \circ Q \subseteq R$
4. $R \subseteq R \circ Q$
5. Q, R accessibility relations for $\triangle$, $\Box$



Lindström's Semantics

Model: $\langle W, Q, \Vdash \rangle$ with

1. Q conversely well–founded & transitive
2. Q accessibility relation for $\triangle$
3. $w \Vdash \Box A \Leftrightarrow x \Vdash A$ whenever $\big|\{u \mid wQuQx\}\big| = \infty$



Soundness and Completeness

Theorem (Lindström):

1. GLT is arithmetically sound and complete with respect to Interpretation #2.
2. GLT is modally sound and complete with respect to Lindström's semantics.

Theorem :
GLT is modally sound and complete with respect to the standard semantics.

Future Work

1. Arithmetical completeness of GLT w.r.t. Interpretation #1
2. Adding $\triangle$ to ILM; arithmetical completeness (w.r.t. Int. #1) of the resulting system

References

Lindström, P.: On Parikh Provability - an Exercise in Modal Logic. Modality Matters: Twenty-Five Essays in Honour of Krister Segerberg. Henrik Lagerlund and Sten Lindström and Rysiek Sliwinski (eds). Uppsala Philosophical Studies 53, pp.279–288, 2006.